Computing nullity and kernel vectors using NF-package: Counterexamples

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Abstract

A computational technique for calculating nullity vectors and kernel vectors, using the new Finsler package, is introduced. As an application, three interesting counterexamples are given. The first counterexample shows that the two distributions Ker_R and $\mathcal{N}_R$ do not coincide. The second shows that the nullity distribution $\mathcal{N}_{P^\circ}$ is not completely integrable. The third shows that the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ is not a sub-distribution of the nullity distribution $\mathcal{N}_{R^\circ}$.

Keywords: Maple program, New Finsler package, Nullity distribution, Kernel distribution.

MSC 2010: 53C60, 53B40, 58B20, 68U05, 83-08.

Introduction

In the applicable examples of Finsler geometry in mathematics, physics and the other branches of science, the calculations are often very tedious to perform. This takes a lot of effort and time. So, we have to find an alternative method to do these calculations. One of the benefits of using computer is the manipulation of the complicated calculations. This enables to study various examples in different dimensions in applications such as field theories (cf., for example, [4]) and physical applications (cf., for example, [5], [7]). The FINSLER package [6] and the new Finsler package [11] are good illustrations of using computer in the applications of Finsler geometry.

In this paper, we use the new Finsler (NF-) package [11] to introduce a computational technique to calculate the components of nullity vectors and kernel vectors. As an application of this method, we construct three interesting counterexamples. The first shows that the kernel distribution Ker_R and the nullity distribution $\mathcal{N}_R$ associated with the h-curvature R of Cartan connection do not coincide, in accordance with [10]. The second proves that the nullity distribution $\mathcal{N}_{P^\circ}$ associated with the hv-curvature $\overset{\circ}{P}$ of Berwald connection is not completely integrable. Finally, the third counterexample shows that the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ associated with the curvature $\mathfrak{R}$ of Barthel connection is not a sub-distribution of the nullity distribution $\mathcal{N}_{R^\circ}$ associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection.

Following the Klein-Grifone approach to Finsler geometry ([1], [2], [3]), let (M, F) be a Finsler space, where F is a Finsler structure defined on an n-dimensional smooth manifold M. Let H(TM) (resp. V(TM)) be the horizontal (resp. vertical) sub-bundle of the bundle TTM. We use the notations R and P for the h-curvature and hv-curvature of Cartan connection respectively. We also use the notations $\overset{\circ}{R}$ and $\overset{\circ}{P}$ for the h-curvature and hv-curvature of Berwald connection respectively. Finally, $\mathfrak{R}$ will denote the curvature of the Cartan non-linear connection (Barthel connection).

1. Nullity and kernel vectors by the NF-package

In this section, we use the New Finsler (NF-) package [11], which is an extended and modified version of [6], to introduce a computational method for the calculation of nullity vectors and kernel vectors.

Definition 1.1. *Let R be the h-curvature tensor of Cartan connection. The nullity space of R at a point $z \in TM$ is the subspace of $H_z(TM)$ defined by*

$$\mathcal{N}_R(z) := \{X \in H_z(TM) : \; R(X,Y)Z = 0, \; \forall Y, Z \in H_z(TM)\}.$$

The dimension of $\mathcal{N}_R(z)$, denoted by $\mu_R(z)$, is the index of nullity of R at z.

If $\mu_R(z)$ is constant, the map $\mathcal{N}_R : z \mapsto \mathcal{N}_R(z)$ defines a distribution $\mathcal{N}_R$ of rank μ_R called nullity distribution of R.

Any vector field belonging to the nullity distribution is called a nullity vector field.

Definition 1.2. *The kernel space* $\mathrm{Ker}_R(z)$ *of the h-curvature* R *at a point* $z \in TM$ *is the subspace of* $H_z(TM)$ *defined by*

$$\mathrm{Ker}_R(z) = \{X \in H_z(TM) : \, R(Y,Z)X = 0, \, \forall \, Y, Z \in H_z(TM)\}.$$

As in Definition 1.1, the map $z \mapsto \mathrm{Ker}_R(z)$ *defines a distribution called the kernel distribution of* R*. Any vector field belonging to the kernel distribution is called a kernel vector field.*

To calculate the nullity vectors and kernel vectors using the NF-package, let us recall some instructions to make the use of this package easier. When we write, for example, N[i,-j] we mean N^i_j, i.e., positive (resp. negative) index means that it is contravariant (resp. covariant). To lower or raise an index by the metric or the inverse metric, just change its sign from positive to negative or vice versa. The command "*tdiff* (N[i,-j], X[k])"means $\partial_k N^i_j$, the command "*tddiff* (N[i,-j], Y[k])"means $\dot{\partial}_k N^i_j$ and the command "*Hdiff* (N[i,-j], X[k])"means $\delta_k N^i_j$. To introduce the definition of a tensor, we use the command "*definetensor*"and to display its components, we use the command "*show*"as will be seen soon.

Now, let $Z \in \mathcal{N}_R$ be a nullity vector. Then, Z can be written locally in the form $Z = Z^i h_i$, where Z^i are the components of the nullity vector Z with respect to the basis $\{h_i\}$ of the horizontal space, where $h_i := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j}$ and N^j_i are the coefficients of Barthel connection; $i, j = 1, ..., n$. The equation $R(Z,X)Y = 0$, $\forall \, X, Y \in H(TM)$, is written locally in the form

$$Z^j R^i_{hjk} = 0.$$

To derive the resulting system from $Z^j R^i_{hjk} = 0$, we first compute the components R^i_{hjk} using the NF-package. Then, we define a new tensor by the command "*definetensor*"as follows:

```
>  definetensor(RCZ[h,-i,-k] = RC[h,-i,-j,-k]*Z[j]);
>  show(RCZ[h,-i,-k]);
```

Putting $RCZ[h,-i,-k] = 0$, we obtain a homogenous system of algebraic equations. Solving this system, we get the components Z^i.

Remark 1.3. It should be noted that we must not use the notation $X = X^i h_i$ nor the notation $Y = Y^i h_i$ for nullity vectors because $RC[h,-i,-j,-k]*X[j]$ and $RC[h,-i,-j,-k]*Y[j]$ mean to Maple $x^j R^h_{ijk}$ and $y^j R^h_{ijk}$ respectively, which both are not the correct expressions for nullity vectors.

In a similar way, we compute the components of a kernel vector. Let $W = W^i h_i \in \mathrm{Ker}_R$, then $R(X,Y)W = 0$, $\forall \, X, Y \in H(TM)$. This locally gives the homogenous system of algebraic equations:

$$W^h R^i_{hjk} = 0.$$

Then by the NF-package, we can define

```
>  definetensor(RCW[h,-j,-k] = RC[h,-i,-j,-k]*W[i]);
>  show(RCW[h,-j,-k]);
```

Putting $RCW[h,-j,-k]=0$ and solving the resulting system, we get the components W^i of the kernel vector W.

2. Applications and counterexamples

In this section, we provide three interesting counterexamples. We perform the computations using the above mentioned technique and the NF-package. We also make use of the technique of simplification of tesor expressions [11].

The nullity distributions associated with Cartan connection are studied in [12]. The following example shows that *the nullity space $\mathcal{N}_R$ of the h-curvature R of Cartan connection and the kernel* Ker_R *do not coincide.*

Example 1

Let $M=\{(x^1,...,x^4)\in\mathbb{R}^4|\,x^2>0\}$, $U=\{(x^1,...,x^4;y^1,...,y^4)\in\mathbb{R}^4\times\mathbb{R}^4 : y^2\neq 0\ ,y^4\neq 0\}\subset TM$. Let F be defined on U by

$$F:=\ (x2^2y1^4+y2^4+y3^4+y4^4)^{1/4}.$$

By Maple program and NF-package we can perform the following calculations.

```
> F0 := sqrt(x2^2*y1^4+y2^4+y3^4+y4^4);
```

$$F0\ :=\ \sqrt{x2^2y1^4+y2^4+y3^4+y4^4}$$

Barthel connection

```
> show(N[i,-j]);
```

$$N^{x1}_{x1}=\frac{1}{3}\frac{y2}{x2}\qquad N^{x1}_{x2}=\frac{1}{3}\frac{y1}{x2}\qquad N^{x2}_{x1}=-\frac{1}{3}\frac{x2y1^3}{y2^2}\qquad N^{x2}_{x2}=\frac{1}{6}\frac{x2y1^4}{y2^3}$$

h-curvature R of Cartan connection

```
> show(RC[h, -i, -j, -k]);
```

$$RC^{x1}_{x2x1x2}=-\frac{1}{18}\frac{3x2^4y1^8+2x2^2y1^4y4^4+2y3^4x2^2y1^4+13x2^2y1^4y2^4+4y2^8+8y3^4y2^4+8y2^4y4^4}{x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RC^{x2}_{x1x1x2}=\frac{1}{18}\frac{\left(x2^4y1^8+2y3^4x2^2y1^4+2x2^2y1^4y4^4+7x2^2y1^4y2^4+8y2^4y4^4+8y3^4y2^4+12y2^8\right)y1^2}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RC^{x1}_{x1x1x2}=\frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}\qquad RC^{x1}_{x3x1x2}=\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3^3}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RC^{x1}_{x4x1x2}=\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4^3}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}\qquad RC^{x2}_{x2x1x2}=-\frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$

$$RC^{x2}_{x3x1x2}=-\frac{1}{18}\frac{y1^3y3^3\left(4y2^4+x2^2y1^4\right)}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}\qquad RC^{x2}_{x4x1x2}=-\frac{1}{18}\frac{y1^3y4^3\left(4y2^4+x2^2y1^4\right)}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RC^{x3}_{x1x1x2}=\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y1^2y3}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}\qquad RC^{x3}_{x2x1x2}=-\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^3}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RC^{x4}_{x1x1x2}=\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4y1^2}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}\qquad RC^{x4}_{x2x1x2}=-\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4y1^3}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

R-**Nullity vectors**

```
> definetensor(RCW[h, -i, -k] = RC[h, -i, -j, -k]*W[j]);
> show(RCW[h, -i, -k]);
```

$$RCW^{x1}_{x2x1} = \frac{1}{18}\frac{\left(3x2^4y1^8+13x2^2y1^4y2^4+2x2^2y1^4y4^4+2y3^4x2^2y1^4+8y2^4y4^4+4y2^8+8y3^4y2^4\right)W^{x2}}{x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RCW^{x1}_{x2x2} = -\frac{1}{18}\frac{\left(3x2^4y1^8+13x2^2y1^4y2^4+2x2^2y1^4y4^4+2y3^4x2^2y1^4+8y2^4y4^4+4y2^8+8y3^4y2^4\right)W^{x1}}{x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RCW^{x2}_{x1x1} = -\frac{1}{18}\frac{\left(x2^4y1^8+7x2^2y1^4y2^4+2y3^4x2^2y1^4+2x2^2y1^4y4^4+12y2^8+8y2^4y4^4+8y3^4y2^4\right)y1^2W^{x2}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCW^{x2}_{x1x2} = \frac{1}{18}\frac{\left(x2^4y1^8+7x2^2y1^4y2^4+2y3^4x2^2y1^4+2x2^2y1^4y4^4+12y2^8+8y2^4y4^4+8y3^4y2^4\right)y1^2W^{x1}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCW^{x1}_{x1x1} = -\frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)W^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3} \qquad RCW^{x1}_{x1x2} = \frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)W^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$

$$RCW^{x1}_{x3x1} = -\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3^3W^{x2}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)} \qquad RCW^{x1}_{x3x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3^3W^{x1}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCW^{x1}_{x4x1} = -\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4^3W^{x2}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)} \qquad RCW^{x1}_{x4x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4^3W^{x1}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCW^{x2}_{x2x1} = \frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)W^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3} \qquad RCW^{x2}_{x2x2} = -\frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)W^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$

$$RCW^{x2}_{x3x1} = \frac{1}{18}\frac{y1^3y3^3\left(4y2^4+x2^2y1^4\right)W^{x2}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)} \qquad RCW^{x2}_{x3x2} = -\frac{1}{18}\frac{y1^3y3^3\left(4y2^4+x2^2y1^4\right)W^{x1}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCW^{x2}_{x4x1} = \frac{1}{18}\frac{y1^3y4^3\left(4y2^4+x2^2y1^4\right)W^{x2}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)} \qquad RCW^{x2}_{x4x2} = -\frac{1}{18}\frac{y1^3y4^3\left(4y2^4+x2^2y1^4\right)W^{x1}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCW^{x3}_{x1x1} = -\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^2W^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3} \qquad RCW^{x3}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^2W^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$

$$RCW^{x3}_{x2x1} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^3W^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4} \qquad RCW^{x3}_{x2x2} = -\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^3W^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RCW^{x4}_{x1x1} = -\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y1^2y4W^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3} \qquad RCW^{x4}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y1^2y4W^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$

$$RCW^{x4}_{x2x1} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4y1^3W^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4} \qquad RCW^{x4}_{x2x2} = -\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4y1^3W^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

Putting $RCW^h_{ij} = 0$, then we have a system of algebraic equations. The NF-package yields the following solution: $W^1 = W^2 = 0, W^3 = s, W^4 = t, ; s, t \in \mathbb{R}$. Then, any nullity vector W has the form

$$W = sh_3 + th_4. \tag{2.1}$$

R-**Kernel vectors**

```
> definetensor(RCZ[h, -j, -k] = RC[h, -i, -j, -k]*Z[i]);
> show(RCZ[h, -j, -k]);
```

$$RCZ^{x1}_{x1x2} = \frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)Z^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}$$
$$-\frac{1}{18}\frac{\left(3x2^4y1^8+13x2^2y1^4y2^4+2x2^2y1^4y4^4+2y3^4x2^2y1^4+8y2^4y4^4+4y2^8+8y3^4y2^4\right)Z^{x2}}{x2^2\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$
$$+\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3^3Z^{x3}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}+\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4^3Z^{x4}}{x2^2y2^3\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCZ^{x2}_{x1x2} = \frac{1}{18}\frac{\left(x2^4y1^8+7x2^2y1^4y2^4+2y3^4x2^2y1^4+2x2^2y1^4y4^4+12y2^8+8y2^4y4^4+8y3^4y2^4\right)y1^2Z^{x1}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$
$$-\frac{1}{9}\frac{y1^3\left(4y2^4+x2^2y1^4\right)Z^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}-\frac{1}{18}\frac{y1^3y3^3\left(4y2^4+x2^2y1^4\right)Z^{x3}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}-\frac{1}{18}\frac{y1^3y4^3\left(4y2^4+x2^2y1^4\right)Z^{x4}}{y2^6\left(x2^2y1^4+y2^4+y3^4+y4^4\right)}$$

$$RCZ^{x3}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^2Z^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}-\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y3y1^3Z^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

$$RCZ^{x4}_{x1x2} = \frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y1^2y4Z^{x1}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^3}-\frac{1}{18}\frac{\left(4y2^4+x2^2y1^4\right)y4y1^3Z^{x2}}{\left(x2^2y1^4+y2^4+y3^4+y4^4\right)y2^4}$$

Putting $RCZ^h_{ij} = 0$, we obtain a system of algebraic equations. The NF-package yields the solution: $Z^1 = \frac{sy_1}{y_2}$, $Z^2 = s$, $Z^3 = t$ and $Z^4 = \frac{s(x_2y_1^4+y_2^4+2y_3^4+2y_4^4)-ty_2y_3^3}{y_2y_4^3}$.

Then, any kernel vector Z should have the form

$$Z = s\left(\frac{y_1}{y_2}h_1 + h_2 + \frac{x_2y_1^4+y_2^4+2y_3^4+2y_4^4}{y_2y_4^3}h_4\right) + t\left(h_3 - \frac{y_3^3}{y_4^3}h_4\right). \tag{2.2}$$

(for simplicity, we have written x_i and y_i instead of x^i and y^i respectively)

Comparing (2.1) and (2.2), we find no values for s and t which make $Z = W$. Consequently, $\mathcal{N}_R$ and Ker_R can not coincide.

In [9] Youssef proved that the nullity distribution $\mathcal{N}_{R^\circ}$ associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection is completely integrable. He conjectured that the nullity distribution $\mathcal{N}_{P^\circ}$ of the hv-curvature $\overset{\circ}{P}$ of Berwald connection is not completely integrable. In the next example, *we show that his conjecture is true.*

Example 2

Let $M = \mathbb{R}^3$, $U = \{(x^1, x^2, x^3; y^1, y^2, y^3) \in \mathbb{R}^3 \times \mathbb{R}^3 : y^1 \neq 0\} \subset TM$. Let F be defined on U by

$$F := \mathrm{e}^{-x1}\left(y2^3 + \mathrm{e}^{-x1x3}y3\,y1^2\right)^{1/3}.$$

By Maple program and NF-package, we can perform the following calculations.

```
> F0 := exp(-2*x1)*(y2^3+exp(-x1*x3)*y3*y1^2)^(2/3);
```

$$F0 := \mathrm{e}^{-2x1}\left(y2^3 + \mathrm{e}^{-x1x3}y3\,y1^2\right)^{2/3}$$

Barthel connection

```
> show(N[i,-j]);
```

$$N^{x1}_{x1} = -\frac{1}{2}(3+x3)\,y1 \qquad N^{x2}_{x1} = -\frac{3}{4}y2 \qquad N^{x2}_{x2} = -\frac{3}{4}y1$$

$$N^{x3}_{x1} = -\frac{3}{4}\frac{y2^3}{y1^2e^{-x1x3}} \qquad N^{x3}_{x2} = \frac{9}{4}\frac{y2^2}{y1e^{-x1x3}} \qquad N^{x3}_{x3} = -y3\,x1$$

hv-curvature $\overset{\circ}{P}$ of Berwald connection

```
> show(PB[h,-i,-j,-k]);
```

$$PB^{x3}_{x1x1x1} = -\frac{9}{2}\frac{y2^3}{y1^4 e^{-x1x3}} \qquad PB^{x3}_{x1x1x2} = \frac{9}{2}\frac{y2^2}{y1^3 e^{-x1x3}}$$

$$PB^{x3}_{x1x2x2} = -\frac{9}{2}\frac{y2}{y1^2 e^{-x1x3}} \qquad PB^{x3}_{x2x2x2} = \frac{9}{2 y1 e^{-x1x3}}$$

$\overset{\circ}{P}$-Nullity vectors

```
> definetensor(PBW[i,-h,-k] = PB[i,-h,-j,-k]*W[j]);
> show(PBW[i,-h,-k]);
```

$$PBW^{x3}_{x1x1} = -\frac{9}{2}\frac{y2^3 W^{x1}}{y1^4 e^{-x1x3}} + \frac{9}{2}\frac{y2^2 W^{x2}}{y1^3 e^{-x1x3}} \qquad PBW^{x3}_{x2x2} = -\frac{9}{2}\frac{y2 W^{x1}}{y1^2 e^{-x1x3}} + \frac{9}{2}\frac{W^{x2}}{y1 e^{-x1x3}}$$

$$PBW^{x3}_{x1x2} = \frac{9}{2}\frac{y2^2 W^{x1}}{y1^3 e^{-x1x3}} - \frac{9}{2}\frac{y2 W^{x2}}{y1^2 e^{-x1x3}} \qquad PBW^{x3}_{x2x1} = \frac{9}{2}\frac{y2^2 W^{x1}}{y1^3 e^{-x1x3}} - \frac{9}{2}\frac{y2 W^{x2}}{y1^2 e^{-x1x3}}$$

Putting $PBW^h_{ij} = 0$, we get a system of algebraic equations. We have two cases: The first case is $y2 = 0$ and the solution in this case is $W^1 = s$, $W^2 = 0$ and $W^3 = t$. Hence, any$\overset{\circ}{P}$-nullity vector is written in the form $W = sh_1 + th_3$. Take two nullity vectors $X, Y \in \mathcal{N}_{P^\circ}$ such that $X = h_1$ and $Y = h_3$. Their Lie bracket $[X, Y] = -\frac{y_1}{2}\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_3}$, which is vertical.
The second case is $y2 \neq 0$ and the solution in this case is $W^1 = s$, $W^2 = \frac{y_2}{y_1}s$ and $W^3 = t$. Then any $\overset{\circ}{P}$-nullity vector is written in the form $W = s(h_1 + \frac{y_2}{y_1}h_2) + th_3$. Let X and Y be the two nullity vectors in $\mathcal{N}_{P^\circ}$ given by $X = h_1 + \frac{y_2}{y_1}h_2$ and $Y = h_3$. By computing their Lie bracket, we find that $[X, Y] = -\frac{y_1}{2}\frac{\partial}{\partial y_1} + y_3\frac{\partial}{\partial y_3}$, which is vertical.
Consequently, in both cases the Lie bracket $[X, Y]$ does not belong to $\mathcal{N}_{P^\circ}$.

Let $\mathcal{N}_{R^\circ}$ and $\mathcal{N}_{\mathfrak{R}}$ be the nullity distributions associated with the h-curvature $\overset{\circ}{R}$ of Berwald connection and the curvature $\mathfrak{R}$ of the Barthel connection respectively. In [9], Youssef proved that $\mathcal{N}_{R^\circ} \subseteq \mathcal{N}_{\mathfrak{R}}$. The following example shows that *the converse is not true: that is $\mathcal{N}_{R^\circ}$ is a proper sub-distribution of $\mathcal{N}_{\mathfrak{R}}$.*

Example 3

Let $M = \mathbb{R}^4$, $U = \{(x^1, \cdots, x^4; y^1, \cdots, y^4) \in \mathbb{R}^4 \times \mathbb{R}^4 : y^2 \neq 0,\ y^4 \neq 0\} \subset TM$. Let F be defined on U by

$$F := \left(e^{-x2} y1 \sqrt[3]{y2^3 + y3^3 + y4^3}\right)^{1/2}.$$

By Maple program and NF-package, we can perform the following calculations.

```
> F0 := exp(-x2)*y1*(y2^3+y3^3+y4^3)^(1/3);
```

$$F0 := e^{-x2} y1 \sqrt[3]{y2^3 + y3^3 + y4^3}$$

Barthel connection

```
> show(N[i,-j]);
```

$$N^{x2}_{x2} = -\frac{1}{4}\frac{4y2^3 + y3^3 + y4^3}{y2^2} \qquad N^{x2}_{x3} = \frac{3}{4}\frac{y3^2}{y2} \qquad N^{x2}_{x4} = \frac{3}{4}\frac{y4^2}{y2}$$

$N^{x3}_{x2} = -\frac{3}{4} y3 \qquad N^{x3}_{x3} = -\frac{3}{4} y2 \qquad N^{x4}_{x2} = -\frac{3}{4} y4 \qquad N^{x4}_{x4} = -\frac{3}{4} y2$

Curvature $\mathfrak{R}$ of the Barthel connection

```
> show(RG[i, -j, -k]);
```

$RG^{x2}_{x2x3} = -\frac{3}{16} \frac{y3^2 (y2^3 + y3^3 + y4^3)}{y2^4} \qquad RG^{x3}_{x2x3} = \frac{3}{16} \frac{y2^3 + y3^3 + y4^3}{y2^2}$

$RG^{x2}_{x2x4} = -\frac{3}{16} \frac{y4^2 (y2^3 + y3^3 + y4^3)}{y2^4} \qquad RG^{x4}_{x2x4} = \frac{3}{16} \frac{y2^3 + y3^3 + y4^3}{y2^2}$

$RG^{x3}_{x3x4} = \frac{9}{16} \frac{y4^2}{y2} \qquad RG^{x4}_{x3x4} = -\frac{9}{16} \frac{y3^2}{y2}$

$\mathfrak{R}$-nullity vectors

```
> definetensor(RGZ[i, -j] = RG[i, -j, -k]*Z[k]);
> show(RGZ[i, -j]);
```

$RGZ^{x2}_{x2} = -\frac{3}{16} \frac{y3^2 (y2^3 + y3^3 + y4^3) Z^{x3}}{y2^4} - \frac{3}{16} \frac{y4^2 (y2^3 + y3^3 + y4^3) Z^{x4}}{y2^4}$

$RGZ^{x2}_{x3} = \frac{3}{16} \frac{y3^2 (y2^3 + y3^3 + y4^3) Z^{x2}}{y2^4} \qquad RGZ^{x2}_{x4} = \frac{3}{16} \frac{y4^2 (y2^3 + y3^3 + y4^3) Z^{x2}}{y2^4}$

$RGZ^{x3}_{x2} = \frac{3}{16} \frac{(y2^3 + y3^3 + y4^3) Z^{x3}}{y2^2} \qquad RGZ^{x3}_{x3} = -\frac{(3y2^3 + y3^3 + y4^3) Z^{x2}}{16 y2^2} + \frac{9 y4^2 Z^{x4}}{16 y2}$

$RGZ^{x3}_{x4} = -\frac{9}{16} \frac{y4^2 Z^{x3}}{y2} \qquad RGZ^{x4}_{x2} = \frac{3}{16} \frac{(y2^3 + y3^3 + y4^3) Z^{x4}}{y2^2}$

$RGZ^{x4}_{x3} = -\frac{9}{16} \frac{y3^2 Z^{x4}}{y2} \qquad RGZ^{x4}_{x4} = -\frac{3}{16} \frac{(y2^3 + y3^3 + y4^3) Z^{x2}}{y2^2} + \frac{9}{16} \frac{y3^2 Z^{x3}}{y2}$

Putting $RGZ^h_i = 0$, we get a system of algebraic equations. In the case where $y2^3 + y3^3 + y4^3 = 0$, we get the solution $Z^1 = t_1$, $Z^2 = t_2$ and $Z^3 = Z^4 = 0$ where $t_1, t_2 \in \mathbb{R}$. Then,

$$Z = t_1 h_1 + t_2 h_2. \tag{2.3}$$

h-curvature $\overset{\circ}{R}$ of Berwald connection:

```
> show(RB[i, -h, -j, -k]);
```

$RB^{x2}_{x2x2x3} = \frac{3}{16} \frac{(y2^3 + 4y4^3 + 4y3^3) y3^2}{y2^5} \qquad RB^{x2}_{x3x2x3} = -\frac{3}{16} \frac{(2y2^3 + 2y4^3 + 5y3^3) y3}{y2^4}$

$RB^{x2}_{x4x2x3} = -\frac{9}{16} \frac{y4^2 y3^2}{y2^4} \qquad RB^{x3}_{x2x2x3} = \frac{3}{16} \frac{y2^3 - 2y3^3 - 2y4^3}{y2^3} \qquad RB^{x3}_{x3x2x3} = \frac{9}{16} \frac{y3^2}{y2^2}$

$RB^{x3}_{x4x2x3} = \frac{9}{16} \frac{y4^2}{y2^2} \qquad RB^{x2}_{x2x2x4} = \frac{3}{16} \frac{(y2^3 + 4y4^3 + 4y3^3) y4^2}{y2^5} \qquad RB^{x2}_{x3x2x4} = -\frac{9}{16} \frac{y4^2 y3^2}{y2^4}$

$RB^{x2}_{x4x2x4} = -\frac{3}{16} \frac{(2y2^3 + 5y4^3 + 2y3^3) y4}{y2^4} \qquad RB^{x4}_{x2x2x4} = \frac{3}{16} \frac{y2^3 - 2y3^3 - 2y4^3}{y2^3}$

$RB^{x4}_{x3x2x4} = \frac{9}{16} \frac{y3^2}{y2^2} \qquad RB^{x4}_{x4x2x4} = \frac{9}{16} \frac{y4^2}{y2^2} \qquad RB^{x3}_{x2x3x4} = -\frac{9}{16} \frac{y4^2}{y2^2}$

$RB^{x3}_{x4x3x4} = \frac{9}{8} \frac{y4}{y2} \qquad RB^{x4}_{x2x3x4} = \frac{9}{16} \frac{y3^2}{y2^2} \qquad RB^{x4}_{x3x3x4} = -\frac{9}{8} \frac{y3}{y2}$

$\overset{\circ}{R}$-nullity vectors

```
> definetensor(RBW[i, -h, -k] = RB[i, -h, -j, -k]*W[j]);
```

```
> show(RBW[i, -h, -k]);
```

$RBW^{x2}_{x2x2} = -\frac{3}{16}\frac{(y2^3+4y4^3+4y3^3)y3^2W^{x3}}{y2^5} - \frac{3}{16}\frac{(y2^3+4y4^3+4y3^3)y4^2W^{x4}}{y2^5}$

$RBW^{x2}_{x2x3} = \frac{3}{16}\frac{(y2^3+4y4^3+4y3^3)W^{x2}y3^2}{y2^5}$ $\quad RBW^{x2}_{x2x4} = \frac{3}{16}\frac{(y2^3+4y4^3+4y3^3)W^{x2}y4^2}{y2^5}$

$RBW^{x2}_{x3x2} = \frac{3}{16}\frac{(2y2^3+2y4^3+5y3^3)y3W^{x3}}{y2^4} + \frac{9}{16}\frac{y4^2y3^2W^{x4}}{y2^4}$ $\quad RBW^{x2}_{x3x4} = -\frac{9}{16}\frac{W^{x2}y4^2y3^2}{y2^4}$

$RBW^{x2}_{x3x3} = -\frac{3}{16}\frac{(2y2^3+2y4^3+5y3^3)W^{x2}y3}{y2^4}$ $\quad RBW^{x2}_{x4x3} = -\frac{9}{16}\frac{W^{x2}y4^2y3^2}{y2^4}$

$RBW^{x2}_{x4x2} = \frac{9}{16}\frac{y4^2y3^2W^{x3}}{y2^4} + \frac{3}{16}\frac{(2y2^3+5y4^3+2y3^3)y4W^{x4}}{y2^4}$ $\quad RBW^{x3}_{x4x4} = \frac{9}{8}\frac{W^{x3}y4}{y2}$

$RBW^{x2}_{x4x4} = -\frac{3}{16}\frac{(2y2^3+5y4^3+2y3^3)W^{x2}y4}{y2^4}$ $\quad RBW^{x3}_{x2x2} = -\frac{3}{16}\frac{(y2^3-2y3^3-2y4^3)W^{x3}}{y2^3}$

$RBW^{x3}_{x2x3} = \frac{3}{16}\frac{(y2^3-2y3^3-2y4^3)W^{x2}}{y2^3} + \frac{9}{16}\frac{W^{x4}y4^2}{y2^2}$ $\quad RBW^{x3}_{x2x4} = -\frac{9}{16}\frac{W^{x3}y4^2}{y2^2}$

$RBW^{x3}_{x3x2} = -\frac{9}{16}\frac{W^{x3}y3^2}{y2^2}$ $\quad RBW^{x3}_{x3x3} = \frac{9}{16}\frac{W^{x2}y3^2}{y2^2}$ $\quad RBW^{x3}_{x4x2} = -\frac{9}{16}\frac{W^{x3}y4^2}{y2^2}$

$RBW^{x3}_{x4x3} = \frac{9}{16}\frac{W^{x2}y4^2}{y2^2} - \frac{9}{8}\frac{y4W^{x4}}{y2}$ $\quad RBW^{x4}_{x2x2} = -\frac{3}{16}\frac{(y2^3-2y3^3-2y4^3)W^{x4}}{y2^3}$

$RBW^{x4}_{x2x3} = -\frac{9}{16}\frac{W^{x4}y3^2}{y2^2}$ $\quad RBW^{x4}_{x2x4} = \frac{3}{16}\frac{(y2^3-2y3^3-2y4^3)W^{x2}}{y2^3} + \frac{9}{16}\frac{W^{x3}y3^2}{y2^2}$

$RBW^{x4}_{x3x2} = -\frac{9}{16}\frac{W^{x4}y3^2}{y2^2}$ $\quad RBW^{x4}_{x3x3} = \frac{9}{8}\frac{W^{x4}y3}{y2}$ $\quad RBW^{x4}_{x4x4} = \frac{9}{16}\frac{W^{x2}y4^2}{y2^2}$

$RBW^{x4}_{x3x4} = \frac{9}{16}\frac{W^{x2}y3^2}{y2^2} - \frac{9}{8}\frac{y3W^{x3}}{y2}$ $\quad RBW^{x4}_{x4x2} = -\frac{9}{16}\frac{W^{x4}y4^2}{y2^2}$

Putting $RBW^h_{ij} = 0$, we obtain a system of algebraic equations. This system has the solution $W^1 = t$, $t \in \mathbb{R}$ and $W^2 = W^3 = W^4 = 0$. Then,

$$W = th_1. \tag{2.4}$$

Consequently, (2.3) and (2.4) lead to $\mathcal{N}_{\mathfrak{R}} \not\subset \mathcal{N}_{R^\circ}$.

3. Conclusion

In this paper, we have mainly achieved two objectives:

• A computational technique for calculating the nullity and kernel vectors, based on the NF-package, has been introduced.

• Using this technique, three counterexamples have been presented: the first shows that the two distributions Ker_R and $\mathcal{N}_R$ do not coincide. The second proves that the nullity distribution $\mathcal{N}_{P^\circ}$ is not completely integrable. The third shows that the nullity distribution $\mathcal{N}_{\mathfrak{R}}$ is not a sub-distribution of $\mathcal{N}_{R^\circ}$.

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